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Hydrodynamic Substitution and Dynamics of Non–Hamiltonian Systems

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Abstract

The possibility of transforming the system of ordinary differential equations to the system of hydrodynamic Euler-type equations with application of hydrodynamic substitution for the Liouville transport equation for phase density equivalent to the original system of equations. Conditions for integrating of Euler-type equations are obtained by modifying Hamilton-Jacobi method for non-Hamiltonian dynamical systems.

Keywords: Hydrodynamic Substitution, Liouville Equation, Hamilton-Jacobi Method

1. INTRODUCTION

The area of applcability of "hydrodynamic substitution" [1, 2], consisting in a special decomposition representation of the solution equations of kinetic type for n-dimensional distribution functions (singling out the dependence of $(n-m)$–dimensional distribution density for an explicitly given area values of m additional to those involved in determining the density of independent variables), until recently it was assumed to be quite limited, basically, the theory of the Vlasov equation. However, in the papers [3, 4] it was found that this substitution also allows us to investigate the properties Liouville equation, revealing its connection both with the general equation of the Hamilton–Jacobi method [5], and with the reduced system of Euler–type hydrodynamic equations (RSE) obtained with its help, and also applicable in the theory of commuting Arnold–Kozlov fields. Moreover, a careful analysis of the properties of the substitution under study shows that it has the property of wide versatility and can be used for the study of systems of ordinary differential equations (ODEs) fairly general. In this paper, we consider the possibility application of hydrodynamic substitution for research systems of autonomous ODEs with subsequent use of the Hamilton–Jacobi method to solve it, moreover, as it turns out, this method (after a proper generalization) is applicable, among other things, to dynamic systems that are not Hamiltonian.

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2. HYDRODYNAMIC SUBSTITUTION FOR A GENERAL SYSTEM OF ODES OF THE 1ST ORDER

We consider an autonomous system of ordinary differential equations of the 1st order in an n-dimensional space:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in R^n, \ \ \mathbf{v}(\mathbf{x}) \in R^n, \ \ t \in R^1. \tag{1}$$

Let define the distribution function $f(\mathbf{x}, t)$ of imaging points in an n-dimensional phase space, which is the probability that the point of the trajectory of the dynamic system is in the neighborhood given point in space R^n at time t. The phase probability transfer is described by the Liouville equation:

$$\frac{\partial f}{\partial t} + \mathrm{div}_n(\mathbf{v}f) = 0. \tag{2}$$

Let us represent the vector $\mathbf{x}$ as an ordered collection $(\mathbf{q}, \mathbf{p})^T$, $\mathbf{q} \in R^m$, $\mathbf{p} \in R^{n-m}$ $(m = 1, 2, ..., n-1)$; in other words, we expand the phase space of the system into a Cartesian sum of phase sets defined by new variables: $R^n_{\mathbf{x}} = R^m_{\mathbf{q}} \oplus R^{n-m}_{\mathbf{p}}$. In this case, system (1) takes the following form:

$$\frac{d\mathbf{q}}{dt} = \mathbf{w}(\mathbf{q}, \mathbf{p}), \quad \frac{d\mathbf{p}}{dt} = \mathbf{g}(\mathbf{q}, \mathbf{p}) \quad \big(\mathbf{w}(\mathbf{q}, \mathbf{p}), \mathbf{g}(\mathbf{q}, \mathbf{p})\big)^T = \mathbf{v}(\mathbf{x}).$$

We will look for a solution to equation (2) using the hydrodynamic substitution:

$$f(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}, t)\delta\big(\mathbf{p} - \mathbf{P}(\mathbf{q}, t)\big),$$

where $\rho(\mathbf{q}, t)$ is some function ("distribution density" in $R^m_{\mathbf{q}} \oplus R^1_t$) to be determined, the set of values of the function $\mathbf{P}(\mathbf{q}, t)$ has the meaning of the functional domain of nontrivial values of the variable $\mathbf{p}$. We assume successively $m = 1, ..., n-1$, then we have: $f = \rho(\mathbf{q}, t) \prod_{k=1}^{n-m} \delta\big(p_k - P_k(\mathbf{q}, t)\big)$. Substituting the last expression into the Liouville equation leads to the system of equations [3]–[4]:

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{m} \frac{\partial}{\partial q_k}\big(\rho W_k\big) = 0, \quad \frac{\partial (\rho \mathbf{P})}{\partial t} + \sum_{k=1}^{m} \frac{\partial}{\partial q_k}\big(\rho W_k \mathbf{P}\big) = \rho \mathbf{G}, \tag{3}$$

$$\mathbf{W}(\mathbf{q}, t) = \mathbf{w}\big(\mathbf{q}, \mathbf{P}(\mathbf{q}, t)\big), \quad \mathbf{G}(\mathbf{q}, t) = \mathbf{g}\big(\mathbf{q}, \mathbf{P}(\mathbf{q}, t)\big).$$

Differentiating in the second equation of the system and taking into account the first, eliminate the dependence on ρ in the second equation. We obtain the equivalence (3) and reduced Euler system of equations:

$$\frac{\partial \rho(\mathbf{q}, t)}{\partial t} + \sum_{k=1}^{m} \frac{\partial}{\partial q_k}\big(\rho W_k(\mathbf{q}, t)\big) = 0, \tag{4}$$

$$\frac{\partial \mathbf{P}(\mathbf{q}, t)}{\partial t} + \sum_{k=1}^{m} W_k \frac{\partial \mathbf{P}(\mathbf{q}, t)}{\partial q_k} = \mathbf{G}(\mathbf{q}, t). \tag{5}$$

Note that the second equation of this system is generalization of the Lamb equation [5] to the case involving the presence non-Hamiltonian forces.

3. TOPOLOGICAL PROPERTIES OF THE EQUATIONS OF RSE AND THEIR SOLUTIONS

Let consider the possibility of applying the hydrodynamic substitution on specific examples, associated with the structure of chaotic hydrodynamic flows.

Example 1. The system of equations for describing the Arnold–Beltrami–Childress flow (ABC) has the form:

$$\dot{x} = A\sin\, z + C\cos\, y, \quad \dot{y} = B\sin\, x + A\cos\, z, \quad \dot{z} = C\sin\, y + B\cos\, x.$$

If we choose $\mathbf{q} = (x)$, $\mathbf{p} = (y, z)^T$, then we have: $\mathbf{W} \equiv \mathbf{w}(\mathbf{q},\mathbf{p})|_{\mathbf{q}=X,\,\mathbf{p}=(Y,Z)} = (A\sin\, Z + C\cos\, Y)$, $\mathbf{G} \equiv \mathbf{g}(\mathbf{q},\mathbf{p})|_{\mathbf{q}=X,\,\mathbf{p}=(Y,Z)} = \big(B\sin\, X + A\cos\, Z,\ C\sin\, Y + B\cos\, X\big)^T$. The RSE equations (4) take the form:

$$\frac{\partial\rho(X,t)}{\partial t} + \frac{\partial\rho}{\partial X}\big(A\sin\, Z + C\cos\, Y\big) = \frac{\partial Y}{\partial X}\rho C\sin\, Y - \frac{\partial Z}{\partial X}\rho A\cos\, Z,$$

$$\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial X}\big(A\sin\, Z + C\cos\, Y\big) = B\sin\, X + A\cos\, Z, \tag{6}$$

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial X}\big(A\sin\, Z + C\cos\, Y\big) = C\sin\, Y + B\cos\, X. \tag{7}$$

This representation allows us to study some properties of the original ABC system. The last two equations (6) and (7) do not depend on the ρ density, so they can be consider separately.

They describe the evolution of an arbitrary line $Y(X,t)$ and $Z(X,t)$ with time due to the Arnold–Beltrami system. Note that this is the meaning of Eq. (5) in the general case: it describes the evolution of m-dimensional hypersurfaces, and its stationary case $\sum W_k \partial\mathbf{P}/\partial q_k = \mathbf{G}$ — equation of invariant stationary surfaces.

Solutions of the ABC-flow system in the stationary case (for $\partial Y/\partial t = \partial Z/\partial t \equiv 0$) are *invariant lines* (time-eliminated trajectories). They satisfy equations (6)–(7), reduced to the following form of two ODEs of the 1st order:

$$\frac{dY}{dX} = \frac{B\sin\, X + A\cos\, Z}{A\sin\, Z + C\cos\, Y}, \quad \frac{dZ}{dX} = \frac{C\sin\, Y + B\cos\, X}{A\sin\, Z + C\cos\, Y}.$$

Example 2. Lorentz's system of equations [6] applied when modeling hydrodynamic turbulence, has the following form:

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy \quad (\sigma, r, b > 0). \tag{8}$$

Choosing $\mathbf{q} = (X)$, $\mathbf{p} = (Y,Z)^T$, we have: $\mathbf{W} \equiv \mathbf{w}(\mathbf{q},\mathbf{p})|_{\mathbf{q}=X,\,\mathbf{p}=(Y,Z)} = (-\sigma X + \sigma Y)$, $\mathbf{G} \equiv \mathbf{g}(\mathbf{q},\mathbf{p})|_{\mathbf{q}=X,\,\mathbf{p}=(Y,Z)} = \big(rX - Y - XZ,\ XY - bZ\big)^T$. Then the system (8) takes the form:

$$\frac{\partial\rho(X,t)}{\partial t} - \frac{\partial\rho}{\partial X}(\sigma X - \sigma Y) = \rho\sigma\left(1 - \frac{\partial Y}{\partial X}\right),$$

$$\frac{\partial Y}{\partial t} - \frac{\partial Y}{\partial X}(\sigma X - \sigma Y) = rX - Y - XZ, \quad \frac{\partial Z}{\partial t} - \frac{\partial Z}{\partial X}(\sigma X - \sigma Y) = XY - bZ. \tag{9}$$

Equations (9) describe the evolution of an arbitrary line $Y(X,t)$ and $Z(X,t)$ with time due to the Lorentz system.

Solutions of the Lorentz system in the stationary case (for $\partial Y/\partial t = \partial Z/\partial t \equiv 0$):

$$\frac{dY}{dX} = \frac{rX - Y - XZ}{-\sigma X + \sigma Y}, \qquad \frac{dZ}{dX} = \frac{XY - bZ}{-\sigma X + \sigma Y}.$$

are invariant lines (trajectories with excluded time).

Thus, we can say that the stationary version (8) describes the *invariant* (relative to the Lorentz system) *surface*, which is *woven from the trajectories of the given system*.

4. SEPARATION OF VARIABLES FOR RSE: THE 2–DIMENSIONAL CASE

In [3], where the possibility of using the hydrodynamic substitution for a general ODE, it is shown that it can be connected in a certain way with the classical Hamilton–Jacobi method.

In fact, equation (5) (which can be considered separately from the continuity equation (4)) is a generalization of the basic equation of the mentioned method for the evolution of the eikonal $S(\mathbf{q},t)$ in the Hamiltonian system $\partial S/\partial t + H(\mathbf{q}, \partial S/\partial \mathbf{q}, t) = 0$, and its stationary version $\mathbf{W}\nabla\mathbf{P} = \mathbf{G}$ — respectively, a generalization of the consequence of the autonomy of the Hamiltonian $H(\mathbf{q}, \partial S/\partial \mathbf{q}) = E(= \mathrm{const})$. At the same time, in particular, one can consider extending this method to "non-symmetrical" phase space ($m \neq n/2$ in general case) of the set of differential equations (1), in which the evolution of a non-Hamiltonian system (described by the generalized Lamb–Kozlov equation).

The main issue in this case is the establishment of criteria for complete integration (i.e. up to quadratures) available dynamic equations. In classical mechanics, to obtain explicit solutions to the Hamilton–Jacobi equation a very common technique for selecting a special coordinate system (CS), in which the equations of motion allow one to separate the variables. Let us find out under what conditions this technique can be used in cases of particular interest, cases of coordinate systems.

Example 3. Let's consider the 2-dimensional case of the system of equations (1):

$$\dot{x} = v_1(x,y), \quad \dot{y} = v_2(x,y). \tag{10}$$

Stationary equation (5) in this case takes the form: $V_1 \cdot dY/dx = V_2$ (here $V_{1,2} = v_{1,2}(x, Y(x))$).

It can be obtained also, if we divide the 2nd equation of the above system into the 1st term by term: $dy/dx = v_2/v_1$.

Hence the condition for the separation of variables is $v_2/v_1 = \alpha(x)\beta(y)$, where $\alpha(x)$ and $\beta(y)$ are arbitrary (integrable) functions of their arguments.

In polar CS $x = \rho\cos(\varphi)$, $y = \rho\sin(\varphi)$, and system of equations (10) becomes following view:

$$\frac{d\rho}{dt} = \widetilde{v}_1(\rho,\varphi)\cdot\cos(\varphi) + \widetilde{v}_2(\rho,\varphi)\cdot\sin(\varphi), \quad \frac{d\varphi}{dt} = \frac{1}{\rho}\big(-\widetilde{v}_1(\rho,\varphi)\cdot\sin(\varphi) + \widetilde{v}_2(\rho,\varphi)\cdot\cos(\varphi)\big), \tag{11}$$

where $\widetilde{v}_{1,2}(\rho,\varphi) = v_{1,2}\big(\rho\cos(\varphi), \rho\sin(\varphi)\big)$. We divide the 1st equation into the 2nd one term by term, we get

$$\frac{d\rho}{d\varphi} = \rho\frac{\dfrac{\widetilde{v}_1}{\widetilde{v}_2}\cos\varphi + \sin\varphi}{-\dfrac{\widetilde{v}_1}{\widetilde{v}_2}\sin\varphi + \cos\varphi}.$$

Thus, the condition for the separation of variables is the condition $\widetilde{v}_1/\widetilde{v}_2 = \gamma(\varphi)$ (where $\gamma(\varphi)$ is an arbitrary function of its argument).

However, this condition is not necessary there are others like it (in a sense, exceptions). Let's demonstrate this using the example of the 2-dimensional Poincare system [7]:

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2), \quad \frac{dy}{dt} = -x + y(1 - x^2 - y^2).$$

Let us pass to polar coordinate system: $d\rho/dt = \rho(1-\rho^2)$, $d\varphi/dt = -1$. So the variables split even though $\widetilde{v}_1/\widetilde{v}_2 = \big(\mathrm{tg}(\varphi) + 1 - \rho^2\big)/\big(-1 + \mathrm{tg}(\varphi)(1-\rho^2)\big) \neq \gamma(\varphi)$ (in this case, in representation (11) one should formally put $\varphi = 0$). For a system of Poincare type $d\rho/dt = \lambda(\rho)$ more general than the one obtained above, $d\varphi/dt = C$ ($\lambda(\rho)$ and C are an arbitrary function and a constant, respectively), equating the right side of (11), considered as a system of linear algebraic equations, vector of right parts $\big(\lambda(\rho), C\big)^T$, we obtain, by solving this system, a new condition for separability of variables in (11):

$$\widetilde{v}_1 = \lambda(\rho)\cos(\varphi) - \rho\sin(\varphi), \quad \widetilde{v}_2 = C\rho\cos(\varphi) + \lambda(\rho)\sin(\varphi).$$

For the elliptic CK (σ, τ) (related to the Cartesian one via the relations $x = \mathrm{ch}(\sigma)\cos(\tau)$, $y = \mathrm{sh}(\sigma)\sin(\tau)$) split variables in the system $\dot{\sigma} = \varkappa_1$, $\dot{\tau} = \varkappa_2$ (for known given functions $\varkappa_1 = \varkappa_1(\sigma,\tau)$, $\varkappa_2 = \varkappa_2(\sigma,\tau)$) occurs under the condition

$$\widetilde{v}_1(\sigma,\tau) = \big(\varkappa_1\mathrm{ch}^2(\sigma)\mathrm{sh}(\sigma)\cos(\tau) + \varkappa_2\mathrm{ch}(\sigma)\sin(\tau)\cos^2(\tau) - \varkappa_2\mathrm{ch}^3(\sigma)\sin(\tau) - \varkappa_1\mathrm{ch}^3(\tau)\mathrm{sh}(\sigma)\big)\times$$

$$\times\big(\sin(\tau)^2\mathrm{ch}^2(\sigma) + \mathrm{sh}^2(\sigma)\sin(t)\cos(t)\big)^{-1},$$

$$\widetilde{v}_2(\sigma,\tau) = \big(\varkappa_1\mathrm{ch}^3(\sigma) - \varkappa_1\mathrm{ch}(\sigma)\cos^2(\tau) - \varkappa_2\cos^2(\tau)\mathrm{sh}(\sigma) + \varkappa_2\mathrm{ch}^2(\sigma)\mathrm{sh}(\sigma)\big)\times$$

$$\times\big(\sin(\tau)^2\mathrm{ch}^2(\sigma) + \mathrm{sh}^2(\sigma)\sin(t)\cos(t)\big)^{-1}.$$

In conclusion, we note that, generally speaking, the equation $V_1(x,Y)dY = V_2(x,Y)dx$ does not have to be solved by dividing variables by choosing an appropriate coordinate system, since it belongs to the class of equations in total differentials (ETD), integrable *always* — in the extreme case, with the involvement of an integrating factor. In particular, with the multiplicative representation $V_k(x,Y) = V_k^{(1)}(x)V_k^{(2)}(Y)$ ($k = 1.2$) the integrating factor has the form $1/\big(V_1^{(1)}(x)V_2^{(2)}(Y)\big)$.

5. SEPARATION OF VARIABLES FOR EQUATIONS OF RSE: 3-DIMENSIONAL CASE

Let us turn to the consideration of 3-dimensional coordinate systems in relation to the system of equations

$$\dot{x} = v_1(x,y,z), \quad \dot{y} = v_2(x,y,z), \quad \dot{z} = v_3(x,y,z). \tag{12}$$

Example 4. We start with the cylindrical coordinate system (ρ, φ, z). Here, three different choices of the dependent (impulse) variable are possible: $Z = Z(\rho, \varphi)$, $R = R(\varphi, z)$, $\Phi = \Phi(\rho, z)$. In connection with their uniformity, we study only the first case. Stationary Equation (5) then takes the form

$$\widehat{V}_1 \frac{\partial Z}{\partial \rho} + \frac{1}{\rho} \widehat{V}_2 \frac{\partial Z}{\partial \varphi} = \widehat{V}_3, \quad \widehat{V}_k = \widehat{v}_k\big(\rho, \varphi, z\big)|_{z=Z(\rho,\varphi)}, \quad k = \overline{1,3}.$$

where $\widehat{v}_k\big(\rho, \varphi, z\big)$ are the right parts of the system of equations (12), converted to the form $\dot{\rho} = \widehat{v}_1$, $\dot{\varphi} = \widehat{v}_2$, $\dot{z} = \widehat{v}_3$ ($\widehat{v}_{1,2}$ coincide with the right-hand sides of equations (11), $\widehat{v}_3 = \widetilde{v}_3(\rho, \varphi, z) \equiv v_3\big(\rho\cos(\varphi), \rho\sin(\varphi), z\big)$). The corresponding system of characteristic equations:

$$\frac{\rho^{-1} d\rho}{\widehat{V}_1} = \frac{d\varphi}{\widehat{V}_2}, \qquad \frac{dZ}{\widehat{V}_3} = \frac{d\rho}{\widehat{V}_1}, \qquad \frac{dZ}{\widehat{V}_3} = \frac{d\varphi}{\rho^{-1}\widehat{V}_2}.$$

From here one can directly obtain six pairs of direct integrability conditions (in quadratures) the above equation of the stationary equation (5):

$$\frac{\widehat{V}_1}{\widehat{V}_2} = \mu_{12}(\varphi),\ \frac{\widehat{V}_1}{\widehat{V}_3} = \mu_{13}(Z); \quad \frac{\widehat{V}_1}{\widehat{V}_2} = \mu_{12}(\varphi),\ \frac{\widehat{V}_2}{\widehat{V}_3} = \mu_{23}(Z); \quad \frac{\widehat{V}_2}{\widehat{V}_3} = \mu_{23}(Z),\ \frac{\widehat{V}_1}{\widehat{V}_3} = \mu_{13}(Z);$$

$$\frac{\widehat{V}_2}{\widehat{V}_1} = \mu_{21}(\rho),\ \frac{\widehat{V}_3}{\widehat{V}_1} = \mu_{31}(\rho); \quad \frac{\widehat{V}_2}{\widehat{V}_1} = \mu_{21}(\rho),\ \frac{\widehat{V}_3}{\widehat{V}_2} = \mu_{32}(\varphi); \quad \frac{\widehat{V}_3}{\widehat{V}_2} = \mu_{32}(\varphi),\ \frac{\widehat{V}_3}{\widehat{V}_1} = \mu_{31}(\rho),$$

where $\mu_{ik}(\ldots)$ are arbitrary (integrable) functions of their argument (possibly degenerating into constants). Apart from the above conditions, you can formulate equally simple sufficient conditions using the fact already noted above universal integrability of 2-dimensional ETD:

$$\widehat{V}_1 = \widehat{V}_1(\rho, \varphi), \quad \widehat{V}_2 = \widehat{V}_2^{(1)}(\rho)\widehat{V}_2^{(2)}(\varphi)\widehat{V}_3^{(1)}(Z), \quad \widehat{V}_3 = \widehat{V}_3(\rho, Z),$$

and also two triples of equalities with the corresponding multiplicative representation already $\widehat{V}_1$ and $\widehat{V}_3$.

Example 5. Now we turn to the consideration of the possibility of using when analyzing system (12) of the spherical coordinate system (r, θ, ϕ). Taking $\Theta = \Theta(r, \phi)$ as the impulse variable and, accordingly, $\mathbf{W} = \big(\widehat{V}_1(r, \Theta, \phi), \widehat{V}_3(r, \Theta, \phi)\big)^T$ and $\mathbf{G} = \widehat{V}_2(r, \Theta, \phi)$, where

$$\widehat{V}_1 = \widetilde{V}_1\cos(\Theta)\sin(\phi) + \widetilde{V}_2\sin(\Theta)\sin(\phi) + \widetilde{V}_3\cos(\phi), \quad \widehat{V}_2 = \frac{1}{r\sin(\phi)}\big(\widetilde{V}_2\cos(\Theta) - \widetilde{V}_1\sin(\ Theta)\big),$$

$$\widehat{V}_3 = \frac{\widetilde{V}_1}{r}\cos(\Theta)\cos(\phi) + \frac{\widetilde{V}_2}{r}\sin(\ Theta)\cos(\phi) - \frac{\widetilde{V}_3}{r}\sin(\phi),$$

we have the stationary equation (5) in the following form:

$$\widehat{V}_1 \frac{\partial \Theta}{\partial r} + \frac{\widehat{V}_3}{r\sin(\Theta)} \frac{\partial \Theta}{\partial \phi} = \widehat{V}_2.$$

As above, we compose the equations of characteristics:

$$\frac{dr}{\widehat{V}_1} = \frac{d\Theta}{\widehat{V}_2}, \qquad \sin(\Theta)\frac{d\phi}{\widehat{V}_3} = \frac{1}{r}\frac{dr}{\widehat{V}_1}, \qquad : \mathrm{cosec}(\Theta)\frac{d\Theta}{\widehat{V}_2} = r\frac{d\phi}{\widehat{V}_3}$$

Therefore, it is easy to obtain direct integrability conditions: 1) $\widehat{V}_1/\widehat{V}_2 = \eta_{12}(\Theta)$, $\widehat{V}_3/\widehat{V}_2 = \eta_{32}(\Theta)$, 2) $\widehat{V}_2/\widehat{V}_3 = \eta_{23}(\phi)$, $\widehat{V}_2/\widehat{V}_1 = \eta_{21}(r)$ etc. d. (here η_{ik} are arbitrary functions of their arguments), as well as sufficient conditions using the corollary of the representation of characteristic equations in the form of a PDE:

$$\widehat{V}_1 = \widehat{V}_1(r,\Theta) = \widehat{V}_1^{(1)}(r)\mathrm{cosec}(\Theta), \quad \widehat{V}_2 = \widehat{V}_2(r,\Theta), \quad \widehat{V}_3 = \widehat{V}_3(r,\phi);$$

$$\widehat{V}_1 = \widehat{V}_1(r,\Theta), \quad \widehat{V}_2 = \widehat{V}_2^{(1)}(r)\widehat{V}_2^{(2)}(\Theta), \quad \widehat{V}_3 = \widehat{V}_2(\Theta,\phi).$$

6. THE POSSIBILITY OF SEPARATION OF VARIABLES IN THE N–DIMENSIONAL CASE FOR CARTESIAN COORDINATES

In this case, the condition for the complete separation of variables, up to the complete finding of the integrals vector equation (we consider the case $m > n - m$)

$$\sum_{i=1}^{m} W_i(\mathbf{q})\frac{\partial \mathbf{P}}{\partial q_i} - \mathbf{G}(\mathbf{q}) = 0 \quad (\dim \mathbf{W} = m, \ \ \dim \mathbf{P} = \dim \mathbf{G} = n - m) \tag{10}$$

in quadratures, is the condition for the multiplicative representation of the vector of right-hand sides of the original ODE system $\mathbf{v}(\mathbf{x})$, and thus the vectors $\mathbf{W}(\mathbf{q})$ and $\mathbf{G}(\mathbf{q})$, t. e.

$$\mathbf{W}(\mathbf{q}) = \Big(\prod_{j=1}^{m} W_1^{(j)}(q_j), ..., \prod_{j=1}^{m} W_m^{(j)}(q_j)\Big)^T, \quad \mathbf{G}(\mathbf{q}) = \Big(\prod_{j=1}^{m} G_1^{(j)}(q_j), ..., \prod_{j=1}^{m} G_{n-m}^{(j)}(q_j)\Big)^T.$$

Indeed, considering for simplicity the scalar version of (10) (i.e. $\mathbf{P} = P_1$, $\mathbf{G} = G_1$), we see that this equation is an inhomogeneous (quasi)linear ODE of the 1st order, which can be reduced to the following homogeneous form (see, for example, [8]):

$$\sum_{i=1}^{m}\prod_{j=1}^{m} W_i^{(j)}(q_j)\frac{d\Omega}{dq_i} + G_1(\mathbf{q})\frac{d\Omega}{dP_1} = 0, \quad G_1(\mathbf{q}) \equiv \prod_{j=1}^{m} G_1^{(j)}(q_j),$$

where $\Omega(\mathbf{q}, P_1)$ is some arbitrary smooth function (satisfying conditions of the implicit function theorem) such that from the equation $\Omega(\mathbf{q}, P_1) = 0$ the dependent variable P_1 is found that satisfies the original inhomogeneous equation. The corresponding characteristic equations are:

$$\frac{dq_1}{W_1^{(1)}(q_1)W_1^{(m)}(q_m)\prod_{i=2}^{m-1} W_1^{(i)}(q_i)} = ... = \frac{dq_m}{W_m^{(m)}(q_m)W_m^{(1)}(q_1)\prod_{i=2}^{m-1} W_m^{(i)}(q_i)} = \frac{dP_1}{\prod_{i=1}^{m} G_1^{(i)}(q_i)}$$

integrated explicitly.

The case $m < n - m$ can be reduced to the previous use for the reduction of the Liouville equation to the system of Euler equations "conjugated" hydrodynamic substitution [3], in which there is a mutual replacement configuration and momenta coordinates.

7. CONCLUSION

In this paper, we consider the possibility of using hydrodynamic substitutions for partial differential equations (hydrodynamic type) obtained from the Liouville equation corresponding to the system of autonomous ordinary differential equations (generally not corresponding to any Hamiltonian systems).

The use of hydrodynamic substitution for the study of differential equations allows you to search for the above solutions on a special given class of functions that have universality and a wide range of applicability. It should be pointed out that the use of the approach under consideration makes it possible to reveal new aspects of the properties of solutions systems of differential equations. It turns out that the hydrodynamic substitution allows a clearer understanding of the essence of the vortex integration method systems of ODEs and is explicitly related to the Hamilton-Jacobi method. In addition, it is possible to generalize the equations of the reduced Euler system to the non-Hamiltonian case, moreover, the Hamilton-Jacobi method is applicable in the latter case as well.

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